

# LEAST-SQUARES LATTICE INTERPOLATION FILTERS

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**ABSTRACT** This paper develops a time as well as order update recursion for linear least-squares lattice (LSL) interpolation filters. The LSL interpolation filter has the nice stage-to-stage modularity which allows its length to be increased or decreased "two-sidedly" (i.e., both *past* and *future*) without affecting the already computed parameters. The LSL interpolation filter is also efficient in computation, flexible in implementation and fast in convergence. The computer simulation results shown in this paper reveal that although interpolation needs more computing power than prediction does, however, interpolation can generate much smaller error power and, thus, reduces much more temporal redundancy than prediction does.

## 1. INTRODUCTION

Linear prediction has many applications in signal processing such as differential pulse code modulation for bandwidth compression, speech processing, adaptive filtering, etc. The performance of these aforementioned applications may be substantially improved by using the less widely known linear interpolation. Linear interpolation is "noncausal" in the sense that the current signal sample can be estimated from a linear combination of its *past* and *future* neighboring samples. The noncausal interpolation filters can be made causal and physically realizable by appending a suitable delay. Some well known theoretical properties and results in linear interpolation based upon minimum mean square error (MMSE) estimation were discussed in [1]-[6]. When data are stationary and statistics are known, the MMSE estimation is a very commonly used criterion. In practice, however, one often has only a finite number of data samples to work with and no knowledge of the data statistics is available. Under the circumstances, the MMSE estimation can no longer be used. The least-squares (LS) estimation can be used to get around this difficulty.

It is widely known that lattice structures have several advantages over their transversal counterparts [9, pp.98]. It was also shown in [1][2][5] that for most processes the minimum mean square interpolation error is smaller than

that of prediction due to the higher correlations between the nearest neighboring samples in the interpolation case. Furthermore, linear interpolation is often better suited to two-dimensional image processing than the linear prediction [7]. These nice features make interpolation lattice realization a good choice for applications such as data compression. This paper develops computationally efficient and fastly convergent recursive LSL filters for linear interpolation.

## 2. LSL INTERPOLATION FILTERS

Let  $\{x(i)\}$ ,  $i = 1, 2, \dots, n$ , be a real discrete-time input signal to a transversal asymmetric interpolation filter of order  $(p, f)$ , where  $n$  is the variable length of the input signal samples. The interpolation filter is *asymmetric* in the sense that the number of past and future signal samples which is  $p$  and  $f$  respectively linearly weighted to estimate the current signal sample  $x(n-f)$  is not necessarily identical [5]. Note that  $x(n)$  is the most recent signal sample used. Let the interpolation coefficient vector at time  $n$  be

$$\mathbf{b}_{p,f}^T(n-f) = [b_{(p,f),f(n-f)}, \dots, b_{(p,f),1(n-f)}, 1, b_{(p,f),-1(n-f)}, \dots, b_{(p,f),-p(n-f)}] \quad (1)$$
 that will be optimized in the least-squares sense over the observation interval  $1-f \leq i \leq n-f$  as follows: Let the  $(q+1)$ -by-1 input vector be given as

$$\mathbf{X}_{q+1}^T(i) = [x(i), x(i-1), \dots, x(i-q)], \quad 1-f \leq i \leq n-f, \quad (2)$$

where  $q$  is the sum of  $p$  and  $f$ . The  $(p, f)$ <sup>th</sup> order interpolation error at each time unit is

$$e_{p,f}^T(i) = \mathbf{b}_{p,f}^T(n-f) \mathbf{X}_{q+1}(i+f), \quad 1-f \leq i \leq n-f. \quad (3)$$

Note that the interpolation coefficients of the interpolation coefficient vector in (1) remain fixed during the observation interval  $1-f \leq i \leq n-f$ . Also note that the use of prewindowing is assumed, that is,  $x(i) = 0$  for  $i \leq 0$ . The optimum interpolation coefficients defined in (1) can be determined by minimizing the sum of the  $(p, f)$ <sup>th</sup> order

interpolation error squares,  $\sum_{i=1-f}^{n-f} (e_{p,f}^T(i))^2$ , with respect to the interpolation coefficients  $b_{(p,f),f(n-f)}, \dots, b_{(p,f),1(n-f)}, b_{(p,f),-1(n-f)}, \dots, b_{(p,f),-p(n-f)}$ . This operation will yield the following deterministic form of the augmented normal

equation for the linear asymmetric interpolation

$$\mathbf{R}_{q+1(n)} \mathbf{b}_{p,f(n-f)} = \mathbf{I}_{p,f(n-f)}, \quad (4)$$

where

$$\mathbf{I}_{p,f(n-f)}^T = \begin{bmatrix} \mathbf{O}_f^T & \mathbf{I}_{p,f(n-f)} & \mathbf{O}_p^T \end{bmatrix},$$

matrix  $\mathbf{R}_{q+1(n)}$  is the  $(q+1)$ -by- $(q+1)$  deterministic correlation matrix and can be expressed as

$$\mathbf{R}_{q+1(n)} = \mathbf{A}_{q+1(x)}^T(n) \mathbf{A}_{q+1(x)}(n), \quad (5)$$

where

$$\mathbf{A}_{q+1(x)}^T(n) = \begin{bmatrix} x(1) & x(2) & \dots & x(n) \\ 0 & x(1) & \dots & x(n-1) \\ 0 & 0 & \dots & x(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & x(n-q) \end{bmatrix} \quad (6)$$

and  $\mathbf{I}_{p,f(n-f)}$  is the minimum value of the sum of the  $(p,f)$ th order interpolation error square with  $x(n)$  being the most recent signal sample used. Vectors  $\mathbf{O}_f$  and  $\mathbf{O}_p$  are column vectors of  $f$  and  $p$  zeros respectively. When  $f$  and  $p$  are set to zero respectively in (4), the deterministic form of the augmented normal equation for the linear asymmetric interpolation will reduce to the following widely known deterministic form of the augmented normal equations for forward and backward predictions:

$$\mathbf{R}_{q+1(n)} \mathbf{a}_q(n) = \begin{bmatrix} \mathbf{P}_{q(n)}^F & \mathbf{O}_q^T \end{bmatrix}^T \quad (7)$$

and

$$\mathbf{R}_{q+1(n)} \mathbf{c}_q(n) = \begin{bmatrix} \mathbf{O}_q^T & \mathbf{P}_{q(n)}^B \end{bmatrix}^T, \quad (8)$$

where

$$\mathbf{a}_q^T(n) = [1, a_{q,1}(n), a_{q,2}(n), \dots, a_{q,q}(n)] \quad (9a)$$

and

$$\mathbf{c}_q^T(n) = [c_{q,q}(n), \dots, c_{q,2}(n), c_{q,1}(n), 1] \quad (9b)$$

are  $q$ th order forward and backward prediction coefficients respectively,  $\mathbf{P}_{q(n)}^F$  and  $\mathbf{P}_{q(n)}^B$  are the minimum value of the sum of the  $q$ th order forward and backward prediction-error squares respectively. Note that efficient order-update recursions used to develop the well known LS prediction lattice called the LSL algorithm was first developed in [11]. We are now prepared to develop the recursive LS asymmetric interpolation lattice filters by embedding the solution of LS interpolation lattice in the solution of LS prediction lattice. We begin the derivation by realizing that the  $(p,f+1)$ st order augmented deterministic interpolation normal equation can be similarly deduced to be

$$\mathbf{R}_{q+2(n)} \mathbf{b}_{p,f+1(n-f-1)} = \begin{bmatrix} \mathbf{O}_{f+1}^T & \mathbf{I}_{p,f+1(n-f-1)} & \mathbf{O}_p^T \end{bmatrix}^T \quad (10)$$

where  $\mathbf{b}_{p,f+1(n-f-1)}^T =$

$[b_{(p,f+1),f+1(n-f-1)}, \dots, b_{(p,f+1),1(n-f-1)}, 1, b_{(p,f+1),-1(n-f-1)}, \dots, b_{(p,f+1),-p(n-f-1)}]$

and the  $(q+2)$ -by- $(q+2)$  deterministic correlation matrix

$$\mathbf{R}_{q+2(n)} = \mathbf{A}_{q+2(x)}^T(n) \mathbf{A}_{q+2(x)}(n). \quad (11)$$

Matrix  $\mathbf{A}_{q+2(x)}^T(n)$  in (11) is defined as

$$\mathbf{A}_{q+2(x)}^T(n) = \begin{bmatrix} x(1) & x(2) & \dots & x(n) \\ 0 & x(1) & \dots & x(n-1) \\ 0 & 0 & \dots & x(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & x(n-q-1) \end{bmatrix} \quad (12)$$

and the scalar,  $\mathbf{I}_{p,f+1(n-f-1)}$ , in (10) is the minimum value of the sum of the  $(p,f+1)$ st order interpolation-error squares with  $x(n)$  being the most recent signal sample used. To

obtain an order-update recursion between  $\mathbf{b}_{p,f+1(n-f-1)}$  and

$\mathbf{b}_{p,f(n-f)}$ , we invert the deterministic correlation matrix

$\mathbf{R}_{q+2(n)}$  in (10)

$$\mathbf{b}_{p,f+1(n-f-1)} = \mathbf{R}_{q+2(n)}^{-1} \begin{bmatrix} \mathbf{O}_{f+1}^T & \mathbf{I}_{p,f+1(n-f-1)} & \mathbf{O}_p^T \end{bmatrix}^T. \quad (13)$$

By defining  $\beta_{p,f+1(n)}^F$  as

$$\mathbf{I}_{p,f+1(n-f-1)} = \beta_{p,f+1(n)}^F \mathbf{I}_{p,f(n-f-1)}, \quad (14)$$

and using the formula [8, pp.577]

$$\mathbf{R}_{q+2(n)}^{-1} = \begin{bmatrix} 0 & \mathbf{O}_{q+1}^T \\ \mathbf{O}_{q+1} & \mathbf{R}_{q+1(n-1)}^{-1} \end{bmatrix} + \frac{\mathbf{a}_{q+1(n)} \mathbf{a}_{q+1(n)}^T}{\mathbf{P}_{q+1(n)}^F},$$

we can recursively obtain the newly updated optimum interpolation coefficient vector  $\mathbf{b}_{p,f+1(n-f-1)}$  from vector

$\mathbf{b}_{p,f(n-f)}$  by using

$$\begin{aligned} \mathbf{b}_{p,f+1(n-f-1)} &= \beta_{p,f+1(n)}^F \left\{ \begin{bmatrix} 0 & \mathbf{O}_{q+1}^T \\ \mathbf{O}_{q+1} & \mathbf{R}_{q+1(n-1)}^{-1} \end{bmatrix} \right. \\ &+ \left. \frac{\mathbf{a}_{q+1(n)} \mathbf{a}_{q+1(n)}^T}{\mathbf{P}_{q+1(n)}^F} \right\} \begin{bmatrix} 0 \\ \mathbf{I}_{p,f(n-f-1)} \end{bmatrix} = \beta_{p,f+1(n)}^F \left\{ \begin{bmatrix} 0 \\ \mathbf{b}_{p,f(n-f-1)} \end{bmatrix} \right. \\ &+ \left. \left( \frac{\mathbf{I}_{p,f(n-f-1)} \mathbf{a}_{q+1(n)} \mathbf{a}_{q+1(n)}^T}{\mathbf{P}_{q+1(n)}^F} \right) \mathbf{a}_{q+1(n)} \right\}. \quad (15) \end{aligned}$$

The ratio  $\beta_{p,f+1(n)}^F$  can be found from the  $(f+2)$ nd row of (15) to be

$$\beta_{p,f+1(n)}^F = \frac{1}{1 + \frac{\mathbf{I}_{p,f(n-f-1)} \mathbf{a}_{q+1(n)} \mathbf{a}_{q+1(n)}^T}{\mathbf{P}_{q+1(n)}^F}}. \quad (16)$$

We will now obtain a lattice structure for the  $(p,f+1)$ st order LS asymmetric interpolation by premultiplying both sides of (15) by row vector  $[x(n), x(n-1), \dots, x(n-q-1)]$ . This yields

$$e_{p,f+1(n-f-1)}^F = \beta_{p,f+1(n)}^F \left\{ e_{p,f(n-f-1)}^F + \left( \frac{\mathbf{I}_{p,f(n-f-1)} \mathbf{a}_{q+1(n)} \mathbf{a}_{q+1(n)}^T}{\mathbf{P}_{q+1(n)}^F} \right) e_{q+1(n)}^F \right\}. \quad (17)$$

The error  $e_{p,f+1(n-f-1)}^F$  represents the interpolation error when one estimates the current signal sample,  $x(n-f-1)$ , from its  $p$  past and  $(f+1)$  future neighboring samples by minimizing the sum of the  $(p,f+1)$ st order interpolation error squares with the most recent signal sample used up to

$x(n)$ . The error  $e_{p,f}^I(n-f-1)$  represents the interpolation error when one estimates the current signal sample,  $x(n-f-1)$ , from its  $p$  past and  $f$  future neighboring samples by minimizing the sum of the  $(p,f)$ <sup>th</sup> order interpolation error squares with the most recent signal sample used up to  $x(n-1)$ . Note that there is one *delay* involved in forming the new order-update interpolation error which will make the LSL interpolation filter physically realizable as one additional *future* signal sample is used. The  $(p+1,f)$ <sup>th</sup> order augmented deterministic interpolation normal equation can be similarly deduced to be

$$\mathbf{R}_{q+2}(n)\mathbf{b}_{p+1,f}(n-f) = \begin{bmatrix} \mathbf{0}_f^T & \mathbf{I}_{p+1,f}(n-f) & \mathbf{0}_{p+1}^T \end{bmatrix}^T, \quad (18)$$

where

$\mathbf{b}_{p+1,f}^T(n-f) = [b_{(p+1,f),(n-f)}, \dots, b_{(p+1,f),1}, b_{(p+1,f),(n-f)}, \dots, b_{(p+1,f),p-1}(n-f)]$  and  $\mathbf{I}_{p+1,f}(n-f)$  is the minimum value of the sum of the  $(p+1,f)$ <sup>th</sup> order interpolation-error squares with  $x(n)$  being the most recent signal sample used. If we define  $\beta_{p+1,f}^B(n)$  as

$$\mathbf{I}_{p+1,f}(n-f) = \beta_{p+1,f}^B(n)\mathbf{I}_{p,f}(n-f), \quad (19)$$

then the following order-update recursion for interpolation error when one additional past signal sample is used to estimate the current signal sample can be similarly obtained:

$$e_{p+1,f}^I(n-f) = \beta_{p+1,f}^B(n) \left\{ e_{p,f}^I(n-f) + \left( \frac{\mathbf{I}_{p,f}(n-f)\mathbf{c}_{q+1,p+1}(n)}{P_{q+1}^B(n)} \right) e_{q+1}^B(n) \right\} \quad (20)$$

where

$$\beta_{p+1,f}^B(n) = - \frac{1}{1 + \frac{\mathbf{I}_{p,f}(n-f)\mathbf{c}_{q+1,p+1}^2(n)}{P_{q+1}^B(n)}}. \quad (21)$$

The error  $e_{p+1,f}^I(n-f)$  represents the interpolation error when one estimates the current signal sample,  $x(n-f)$ , from its  $(p+1)$  past and  $f$  future neighboring samples by minimizing the sum of the  $(p+1,f)$ <sup>th</sup> order interpolation error squares with the most recent signal sample used up to  $x(n)$ . Note that there is no delay involved in forming the new order-update interpolation error as one additional past signal sample is used. Equations (14), (16), (17), (19), (20), and (21) make up the LS asymmetric interpolation lattice solution. The LSL algorithm which is used as the preprocessing operation of this solution can be seen in [8, pp.619]. A single stage LSL realization of  $(p,f+1)$ <sup>st</sup> order and  $(p+1,f)$ <sup>th</sup> order interpolation error filter using (17) and (20) is shown in Figure 1. There are  $q!/p!f!$  different lattice implementations for a  $(p,f)$ <sup>th</sup> order LS interpolation error filter due to an orthogonal basis set discovered in [10][12], where  $q=p+f$ . All the implementations generated by this orthogonal basis set will give the same result.

### 3. SIMULATION RESULTS

In this section we describe a computer simulation experiment which compares the performances of interpolation lattice and prediction lattice. For the purpose of comparison, we use a second order autoregressive process, AR(2), which is defined as  $x(n) + a_1x(n-1) + a_2x(n-2) = \varepsilon(n)$ , where the driving process,  $\varepsilon(n)$ , is a computer generated sequence which simulates a zero-mean Gaussian white noise process with variance  $\sigma_\varepsilon^2$ . The AR parameters  $a_1$  and  $a_2$  are chosen so that the AR process  $x(n)$  has unity variance. For convenience, the AR parameter values closely follow those in [8, pp.286]. Figures 2 and 3 show the results of computer simulations of learning curves by using both LSL forward prediction filters and LSL interpolation filters with the eigenvalue spread of the AR(2), being set to 10 and 100 respectively. The symbols  $\langle (e_{2,2}^I(n))^2 \rangle$ ,  $\langle (e_2^F(n))^2 \rangle$ , and  $\langle (e_4^F(n))^2 \rangle$  are used to denote the 400 ensemble-averaged squared errors for the (2,2)<sup>th</sup> - order interpolation, 2<sup>nd</sup> - order, and 4<sup>th</sup> - order forward predictions respectively. The results of simulations used a  $\log_{10}$  scale. Figures 2 and 3 show that interpolation indeed provides smaller average squared errors (by about 7 dB for this particular AR(2) process) than prediction does. This simply displays that the LSL interpolation, which makes better use of correlation between the nearest neighboring samples than prediction does, can achieve better performance than that of LSL prediction. The computer simulation results also reveal that the learning curves of the LSL interpolation filters display very fast convergence.

### 4. CONCLUSIONS

The development of the LSL interpolation filters in this paper utilizes an *orthogonal basis* consisting of both forward and backward prediction errors first discovered in [10][12]. As a result, the LSL interpolation filter has the nice *decoupling* (or *order-recursive*) property which allows the most effective and efficient expansion and contraction of its length. Furthermore, The LSL interpolation filter is efficient in computation and flexible in implementation.

The simulation results reveal that although interpolation needs more computing power than prediction does, however, interpolation generates much smaller error power and thus reduces much more temporal redundancy than prediction does. As a result, a higher degree of data compression can be achieved by interpolation than by prediction.

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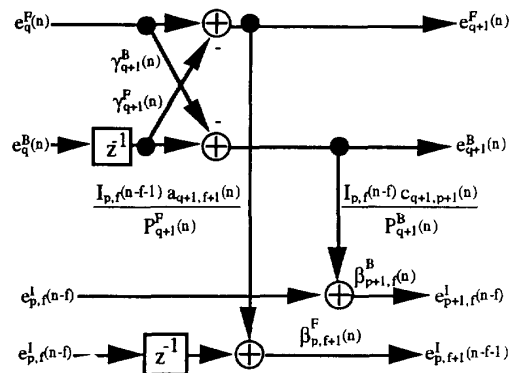


Figure 1. A single stage LSL order-recursive as well as time-recursive interpolation error filter realization of orders  $(p,f+1)$  and  $(p+1,f)$ .

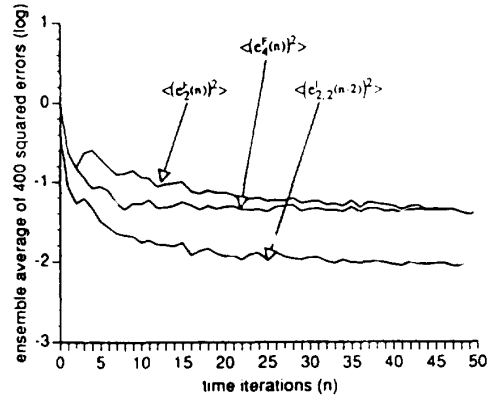


Figure 2. Learning curves ( $\log_{10}$  scale) of the LSL interpolation filter and LSL forward prediction filters with the eigenvalue spread of the AR(2) being set to 10.

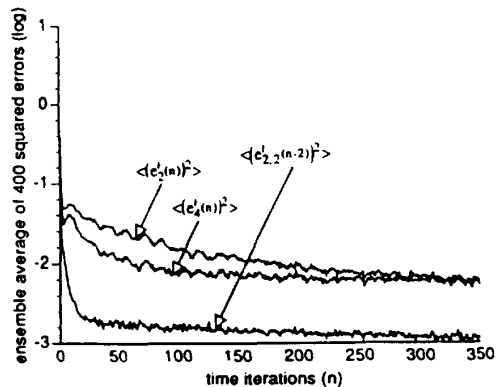


Figure 3. Learning curves ( $\log_{10}$  scale) of the LSL interpolation filter and LSL forward prediction filters with the eigenvalue spread of the AR(2) being set to 100.