A NEW INTERPOLATION LATTICE STRUCTURE

Jenq-Tay Yuan

University of Missouri - Rolla, Rolla, MO 65401

ABSTRACT

This paper presents a new lattice structure for linear interpolation. This interpolation lattice is *asymmetric* in the sense that the number of past and future values linearly weighted to estimate the current value is not necessarily identical. The asymmetric interpolation lattice is computationally efficient and flexible to implement. It is also a generalization of the well-known linear prediction lattice and symmetric interpolation lattice.

INTRODUCTION

Linear interpolation has many applications in signal processing. Some well-known theoretical properties and results in linear interpolation were discussed in [1]-[3]. Fast algorithms for an FIR interpolation filter with linear phase was developed in [4]. A lattice structure for the symmetric interpolation filter was first developed in [5]. The interpolation lattice structure developed in this paper is for stationary random processes using a minimum mean square error (MMSE) criterion and it generalizes some results in [2][3][5] . It is asymmetric in the sense that the number of past and future values linearly weighted to estimate the current value is not necessarily identical. The asymmetric interpolation lattice structure reduces to the well-known linear prediction lattice [6]-[9] when no future signal samples are used to estimate the current signal sample. The interpolation lattice becomes symmetric when an equal number of past and future signal samples are used to estimate the current signal sample.

ASYMMETRIC INTERPOLATION LATTICE

Let x(n) denote the real, wide sense stationary random process taken at the n-th sample. The problem of (p,f)-th order linear asymmetric interpolation can be defined as follows: An estimate of the current signal sample x(n) is to be obtained by linearly weighting p previous and f future signal samples, where p and f are both positive integers. The estimate of the signal sample x(n), denoted by $\widehat{x_{p,f}(n)}$, is given by

$$\widehat{x_{p,f}(n)} = -\sum_{\substack{i=-p \\ i\neq 0}}^{f} b_{(p,f),i}\, x(n+i) \quad , \qquad (1)$$

where $b_{(p,f),i}$ is the interpolation coefficient of order (p,f). The (p,f)-th order linear interpolation error, denoted by $e^I_{p,f}(n)$, is given by

$$e^I_{p,f}(n) = x(n) - \widehat{x_{p,f}(n)} = \sum_{i=-p}^{f} b_{(p,f),i}\, x(n+i) \quad , \qquad (2)$$

where $b_{(p,f),0}$ is defined as unity. We note that the interpolation problem reduces to the forward (or backward) prediction problem upon setting f (or p) equal to zero. We choose the interpolation coefficients in (1) to minimize the mean square interpolation error $E\{(e^I_{p,f}(n))^2\}$. The minimization can be accomplished by using the orthogonality principle [6][7] which results in the following augmented asymmetric interpolation normal equation

$$R_q\, b_{p,f} = i_{p,f} \qquad (3)$$

where $b_{p,f}^T = [b_{(p,f),f},\ldots,b_{(p,f),1},1,b_{(p,f),-1},\ldots,b_{(p,f),-p}]$, $i_{p,f}^T = [0,\ldots,0,I_{p,f},0,\ldots,0]$, $q = p + f$, and $I_{p,f}$ is the minimum mean square interpolation error of order (p,f). The matrix R_q in (3) is the q-th order Toeplitz autocorrelation matrix of the signal sample x(n). Note that there are f zeros above and p zeros below $I_{p,f}$ in vector $i_{p,f}$. We also note that the asymmetric normal equation (3) reduces to the well-known forward (or backward) normal equation of linear prediction theory upon setting f (or p) to zero.

Our objective is to express the interpolation error $e^I_{p,f}(n)$ as a linear combination of the mutually orthogonal forward prediction errors $\{e^F_q(n+f),e^F_{q-1}(n+f-1),\ldots,e^F_0(n-p)\}$. To accomplish this, we first linearly transform the data samples $\{x(n+f),x(n+f-1),\ldots,x(n-p+1),x(n-p)\}$ to the mutually orthogonal forward prediction errors as follows:

$$e^F_q(n) = U_q\, x_{p,f}(n), \qquad (4)$$

The author would like to thank Dr. John A. Stuller for his invaluable suggestions. Dr. Yuan is currently with Formosa Plastics Co.,Texas.

0-7803-0946-4/93 $3.00

where $x_{p,f}(n)^T = [x(n+f), x(n+f-1), \ldots, x(n-p)]$,

and $e_q^F(n)^T = [e_q^F(n+f), e_{q-1}^F(n+f-1), \ldots, e_0^F(n-p)]$.

The matrix U_q in (4) is the (q+1)-by-(q+1) upper-triangular matrix with 1's on the diagonal in the UL Cholesky factorization of the q-th order autocorrelation matrix R_q, given by

$$R_q^{-1} = U_q^T (P_q^R)^{-1} U_q, \qquad (5)$$

where the matrix P_q^R is diagonal with the minimum mean square prediction errors of orders q,q-1,...,0 on the main diagonal and the superscript ^{T} denotes the tranpose operation. The vector $b_{p,f}$ in (3) can be found by substituting R_q^{-1} in (5) into

$$b_{p,f} = R_q^{-1} i_{p,f}. \qquad (6)$$

By using (5) and (6), we express $b_{p,f}$ as

$$b_{p,f} = R_q^{-1} i_{p,f} = \left\{U_q^T (P_q^R)^{-1} U_q\right\} i_{p,f}. \qquad (7)$$

From (2) the (p,f)-th order interpolation error $e_{p,f}^I(n)$ can be written in vector form as

$$e_{p,f}^I(n) = b_{p,f}^T x_{p,f}(n). \qquad (8)$$

By substituting vectors $b_{p,f}$ and $x_{p,f}(n)$ into (8) from (7) and (4) respectively, we obtain

$$e_{p,f}^I(n) = I_{p,f}\{\left(\frac{1}{P_{q-f}}\right) e_{q-f}^F(n) + \left(\frac{a_{q-f+1,1}}{P_{q-f+1}}\right) e_{q-f+1}^F(n+1) + \ldots + \left(\frac{a_{q-1,f-1}}{P_{q-1}}\right) e_{q-1}^F(n+f-1) + \left(\frac{a_{q,f}}{P_q}\right) e_q^F(n+f)\}, \qquad (9)$$

where $a_{m,i}$ is the prediction coefficient of order m and P_m is the MMS prediction error of order m. By introducing f delays, we have

$$e_{p,f}^I(n-f) = I_{p,f}\{\left(\frac{1}{P_p}\right) e_p^F(n-f) + \left(\frac{a_{p+1,1}}{P_{p+1}}\right) e_{p+1}^F(n-f+1) + \ldots + \left(\frac{a_{q-1,f-1}}{P_{q-1}}\right) e_{q-1}^F(n-1) + \left(\frac{a_{q,f}}{P_q}\right) e_q^F(n)\}. \qquad (10)$$

Equation (10) is the result sought. It expresses the (p,f)-th order interpolation error used in estimating data sample x(n-f) as a linear combination of (f+1) mutually orthogonal forward prediction errors. Equation (10) has a nice interpretation. We separate the right side of (10) into two bracketed terms as shown below

$$e_{p,f}^I(n-f) = I_{p,f}\{[\left(\frac{1}{P_p}\right) e_p^F(n-f)] + [\left(\frac{a_{p+1,1}}{P_{p+1}}\right) e_{p+1}^F(n-f+1) + \ldots + \left(\frac{a_{q-1,f-1}}{P_{q-1}}\right) e_{q-1}^F(n-1) + \left(\frac{a_{q,f}}{P_q}\right) e_q^F(n)]\}. \qquad (11)$$

The p-th order forward prediction error $e_p^F(n-f)$ in the first bracket represents the error of estimating data sample x(n-f) based on all its p previous samples. The prediction error $e_p^F(n-f)$ can be improved by using f data samples subsequent to the sample x(n-f). These f data samples corresponding to those f forward prediction errors in the second bracket of (11) can be thought of as a correction term adding some improvements to the prediction error $e_p^F(n-f)$ in the first bracket of (11).

Since all the forward prediction errors in (10) are mutually orthogonal, the (p,f)-th order MMS interpolation error $I_{p,f}$ can be obtained:

$$I_{p,f} = \frac{1}{\left(\frac{1}{P_p} + \frac{a_{p+1,1}^2}{P_{p+1}} + \ldots + \frac{a_{q-1,f-1}^2}{P_{q-1}} + \frac{a_{q,f}^2}{P_q}\right)} \qquad (12)$$

The interpolation error $e_{p,f}^I(n)$ can also be similarly expressed in terms of a linear combination of the mutually orthogonal backward prediction errors by using LU Cholesky factorization, that is,

$$e_q^B(n+f) = L_q x_{p,f}(n),$$

where $e_q^B(n+f)^T = [e_0^B(n+f), e_1^B(n+f), \ldots, e_q^B(n+f)]$,

and L_q is lower triangular with 1's on the diagonal. The interpolation error can be computed to be

$$e_{p,f}^I(n) = I_{p,f}\{\left(\frac{1}{P_f}\right) e_f^B(n+f) + \left(\frac{a_{f+1,1}}{P_{f+1}}\right) e_{f+1}^B(n+f) + \ldots + \left(\frac{a_{q-1,p-1}}{P_{q-1}}\right) e_{q-1}^B(n+f) + \left(\frac{a_{q,p}}{P_q}\right) e_q^B(n+f)\}.$$

By introducing f delays, we have

$$e_{p,f}^I(n-f) = I_{p,f}\{\left(\frac{1}{P_f}\right) e_f^B(n) + \left(\frac{a_{f+1,1}}{P_{f+1}}\right) e_{f+1}^B(n) + \ldots + \left(\frac{a_{q-1,p-1}}{P_{q-1}}\right) e_{q-1}^B(n) + \left(\frac{a_{q,p}}{P_q}\right) e_q^B(n)\}. \qquad (13)$$

The (p,f)-th order MMS interpolation error can also easily be found to be

$$I_{p,f} = \frac{1}{\left(\frac{1}{P_f} + \frac{a_{f+1,1}^2}{P_{f+1}} + \ldots + \frac{a_{q-1,p-1}^2}{P_{q-1}} + \frac{a_{q,p}^2}{P_q}\right)}. \qquad (14)$$

Both equation pairs (10) and (12) and (13) and (14), termed the *asymmetric nonrecursive interpolation lattice* solution, allows us to compute the interpolation error of order (p,f) nonrecursively from the results of the q=(p+f)-th order prediction. The result computed by using (10) or (13) will be exactly the same.

The derivation of the asymmetric *order-recursive* interpolation lattice solution can be proceeded as follows: Similar to the derivation of (9), the (p,f+1)-st order interpolation error $e_{p,f+1}^I(n)$ can be expressed in terms of forward prediction errors as

$$e_{p,f+1}^I(n) = I_{p,f+1}\{\left(\frac{a_{q+1,f+1}}{P_{q+1}}\right) e_{q+1}^F(n+f+1) + [\left(\frac{a_{q,f}}{P_q}\right) e_q^F(n+f) + \ldots + \left(\frac{a_{q-f+1,1}}{P_{q-f+1}}\right) e_{q-f+1}^F(n+1) + \left(\frac{1}{P_{q-f}}\right) e_{q-f}^F(n)]\},$$

where $I_{p,f+1}$ is the minimum mean square interpolation error of order (p,f+1). If we relate $I_{p,f+1}$ and $I_{p,f}$ as

$$I_{p,f+1} = \beta^F_{p,f+1}\, I_{p,f} \tag{15}$$

then we have

$$e^I_{p,f+1}(n) = \beta^F_{p,f+1}\left\{\left(\frac{a_{q+1,f+1}I_{p,f}}{P_{q+1}}\right)e^F_{q+1}(n+f+1) + \left[\left(\frac{a_{q,f}I_{p,f}}{P_q}\right)e^F_q(n+f) + \dots + \left(\frac{I_{p,f}}{P_{q-f}}\right)e^F_{q-f}(n)\right]\right\}, \tag{16}$$

where $\beta^F_{p,f+1}$ is a constant. The bracketed term in equation (16) is identical to $e^I_{p,f}(n)$ shown in (9). Therefore (16) reduces to

$$e^I_{p,f+1}(n) = \beta^F_{p,f+1}\left\{e^I_{p,f}(n) + \left(\frac{a_{q+1,f+1}I_{p,f}}{P_{q+1}}\right)e^F_{q+1}(n+f+1)\right\}.$$

By introducing (f+1) delays, we have

$$e^I_{p,f+1}(n-f-1) = \beta^F_{p,f+1}\left\{e^I_{p,f}(n-f-1) + \left(\frac{a_{q+1,f+1}I_{p,f}}{P_{q+1}}\right)e^F_{q+1}(n)\right\} \tag{17}$$

The quantity $e^I_{p,f+1}(n-f-1)$ in (17) is the (p,f+1)-st order interpolation error in estimating the data sample x(n-f-1) by using its p past data samples x(n-f-2),...,x(n-q-1) and its f+1 future data samples x(n-f),...,x(n). Note that x(n) is the most recent data sample being used to estimate the sample being interpolated. An expression for the constant $\beta^F_{p,f+1}$ in (17) can be obtained by realizing that $e^I_{p,f}(n-f-1)$ and $e^F_{q+1}(n)$ in (17) are orthogonal to each other which can be seen from (16). If we square both sides of (17) and take the expected value, we obtain

$$\beta^F_{p,f+1} = \frac{1}{1 + \dfrac{I_{p,f}\, a^2_{q+1,f+1}}{P_{q+1}}} \tag{18}$$

with the aid of (15). Note that $\beta^F_{p,f+1}$ has a value between zero and one. This implies, by (15), that the MMS interpolation error will decrease as the length of the filter is increased. Equations (15) and (18) provide an order-update recursion for the MMS interpolation error as one more future data sample is weighted to estimate the present data sample. In a similar manner, as one more past data sample x(n-p-1) is linearly weighted to estimate the present data sample x(n), the (p+1,f)-th order interpolation error can be expressed recursively as

$$e^I_{p+1,f}(n) = \beta^B_{p+1,f}\left\{e^I_{p,f}(n) + \left(\frac{a_{q+1,p+1}I_{p,f}}{P_{q+1}}\right)e^B_{q+1}(n+f)\right\}$$

The *delayed* interpolation error is

$$e^I_{p+1,f}(n-f) = \beta^B_{p+1,f}\left\{e^I_{p,f}(n-f) + \left(\frac{a_{q+1,p+1}I_{p,f}}{P_{q+1}}\right)e^B_{q+1}(n)\right\}, \tag{19}$$

where $\beta^B_{p+1,f}$ which is defined to be

$$I_{p+1,f} = \beta^B_{p+1,f}\, I_{p,f} \tag{20}$$

can be found to be

$$\beta^B_{p+1,f} = \frac{1}{1 + \dfrac{I_{p,f}\, a^2_{q+1,p+1}}{P_{q+1}}} \tag{21}$$

The quantity $e^I_{p+1,f}(n-f)$ in (19) is the (p+1,f)-th order interpolation error of estimating x(n-f) by using its p+1 past data samples and its future data samples with x(n) being the most recent data sample being used. Note that $\beta^B_{p+1,f}$ is also greater than zero and less than unity. Equations (20) and (21) provide an order update recursion for the MMS interpolation error as one more past data sample is weighted to estimate the present data sample. Equations (15), (17), (18), (19), (20), and (21) provide order-recursive updates for interpolation error and constitute the *asymmetric order-recursive interpolation lattice* solution. An illustration of this interpolation lattice solution is given by Figure 1, which depicts a single stage lattice realization of the (p,f+1)-st and the (p+1,f)-th order-updated interpolation error from the current (p,f)-th order interpolation error. As illustrated by Figure 1, equations (17) and (19) require that a higher order forward prediction error be used to update the current interpolation error as one more future data sample is used, and a higher order backward prediction error be used as one more past data sample is used. Hence, the use of future data corresponds to updating of the interpolation error with a forward prediction error, and the use of a past data corresponds to updating of interpolation error by using a backward prediction error. A combined use of both (17) and (19) will show that an interpolation operation of any order (p,f) can be expressed in terms of mutually orthogonal forward and backward prediction errors. According to [10] [11], there are q!/p!f! different ways to implement this interpolation error filter. All the implementations will have the same result.

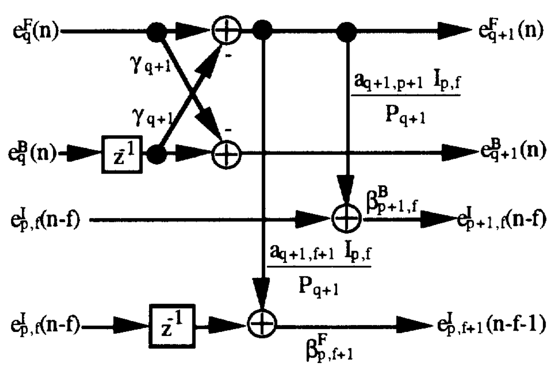

Figure 1. A single stage order-recursive interpolation error filter of order (p,f+1) and (p+1,f).

Both asymmetric order-recursive and nonrecursive lattice interpolation solutions provide a computationally efficient solution to the (p,f)-th order linear interpolation problem once the q-th order linear prediction problem is obtained. For the case of symmetric interpolation of order (p,p), once the results of the 2p-th order linear prediction are known, to solve the interpolation problem, the computational complexity of this asymmetric interpolation solution requires O(p) operations which is in contrast to $O(p^2)$ needed by the algorithm developed in [5].

THREE SPECIAL CASES

The asymmetric interpolation solution developed above can be viewed as a generalization of both prediction and symmetric interpolation and can be easily tailored to fit into the following three special cases.

Forward prediction case: $q = p$, $(p,f) = (p,0)$. It can be shown by using (13) that the p-th order forward prediction error can be expressed as

$$e_p^F(n) = P_p \left\{ \left(\frac{1}{P_0}\right) e_0^B(n) + \left(\frac{a_{1,1}}{P_1}\right) e_1^B(n) + \ldots + \left(\frac{a_{p,p}}{P_p}\right) e_p^B(n) \right\}$$

and the power ratio constant $\beta_{p+1,0}^B$ can also be shown by using (20), (21), and $I_{p+1,0} = P_{p+1}$ to be related to the (p+1)-st order reflection coefficient γ_{p+1} by

$$\beta_{p+1,0}^B = 1 - \gamma_{p+1}^2.$$

This result agrees with the well-known recursion for the MMS prediction errors.

Backward prediction case: $q = f$, $(p,f) = (0,f)$. We can similarly find

$$e_f^B(n) = P_f \left\{ \left(\frac{1}{P_0}\right) e_0^F(n-f) + \left(\frac{a_{1,1}}{P_1}\right) e_1^F(n-f+1) + \ldots + \left(\frac{a_{f,f}}{P_f}\right) e_f^F(n) \right\}$$

and

$$\beta_{0,f+1}^F = 1 - \gamma_{f+1}^2.$$

From the above special cases we see that asymmetric interpolation enables us to view both forward and backward prediction from a broader perspective.

Symmetric interpolation case: $q = 2p$, $(p,f) = (p,p)$. By (10) and (13), the (p,p)-th order interpolation error $e_{p,p}^I(n-p)$ can be written as

$$e_{p,p}^I(n-p) = I_{p,p} \left\{ \left(\frac{a_{2p,p}}{P_{2p}}\right) e_{2p}^F(n) + \left(\frac{a_{2p-1,p-1}}{P_{2p-1}}\right) e_{2p-1}^F(n-1) + \ldots + \left(\frac{1}{P_p}\right) e_p^F(n-p) \right\} \tag{22}$$

and

$$e_{p,p}^I(n-p) = I_{p,p} \left\{ \left(\frac{a_{2p,p}}{P_{2p}}\right) e_{2p}^B(n) + \left(\frac{a_{2p-1,p-1}}{P_{2p-1}}\right) e_{2p-1}^B(n) + \ldots + \left(\frac{1}{P_p}\right) e_p^B(n) \right\} \tag{23}$$

respectively. In this case, the MMSE interpolation filter will naturally have a *linear phase*.

The well-known result of symmetric interpolation error power

$$I_{p,p} = \frac{P_p}{1 + \sum_{i=1}^{p} a_{p,i}^2}$$

shown in [2] for autoregressive process can also be easily derived by using (12), (14).

CONCLUSIONS

In this paper we have developed a new asymmetric interpolation lattice structure. The structure is efficient in computation and flexible in implementation. It also forms a bridge between forward prediction, backward prediction and symmetric interpolation which can all be viewed as special cases of asymmetric interpolation. As a result, the asymmetric interpolation lattice provides a broader interpretation and a more thorough understanding of the linear prediction and linear interpolation theories.

The interpolation lattice structure developed in this paper may be useful in the field of data compression. Although interpolation needs more computing power than prediction does, it reduces more temporal redundancy. As a result, a higher degree of data compression can be achieved by interpolation than by prediction.

REFERENCES

[1] Picinobono, Bernard, and Jean-Marc Kerilis, "Some Properties of Prediction and Interpolation Errors," IEEE Trans. on Acoustics, Speech, and Signal Processing, Vol. ASSP-36, No.4, April,1988.

[2] Kay, Steven, "Some Results in Linear Interpolation Theory," IEEE Trans. on Acoustics, Speech, and Signal Processing, Vol. ASSP-31, No. 3, pp. 746-749, June 1983.

[3] Stuller, John A., "On the Relation Between Triangular Matrix Decomposition and Linear Interpolation," Proceedings of the IEEE, Vol. 72, No. 8, pp.1093-1094, August 1984.

[4] Marple, S. Lawrence Jr., "Fast Algorithms for Linear Prediction and System Identification Filters with Linear Phase," IEEE Trans. on Acoustics, Speech, and Signal Processing, Vol. ASSP-30, No. 6, pp.942-953, December 1982.

[5] Coursey, Cameron K., and John A. Stuller, "Interpolation Lattice Filter," IEEE Trans. on Acoustics, Speech, and Signal Processing, Vol. ASSP-39, No. 4, pp. 965-967, April 1991.

[6] Orfanidis, Sophocles J., "Optimum Signal Processing," New York, New York: Macmillan Publishing Co., 1985.

[7] Giordano, Arthur A., and Frank M. Hsu, "Least Squares Estimation with Applications to Digital Signal Processing," New York, New York: John Wiley and Sons, Inc., 1985.

[8] Yuan, Jenq-Tay, and John A. Stuller, "Order-Recursive FIR Smoothers," submitted for publication in IEEE Trans. on Signal Processing.

[9] Yuan, Jenq-Tay, "Lattice Structures for Noncausal Filters," Ph.D. dissertation, Department of Electrical Engineering, University of Missouri-Rolla, Rolla, Mo., 1991.